

UPPER AND LOWER BOUNDS FOR NUMERICAL RADII OF BLOCK SHIFTS

HWA-LONG GAU AND PEI YUAN WU*

Dedicated to Professor Heydar Radjavi on his 80th birthday

ABSTRACT. For any n -by- n matrix A of the form

$$\begin{bmatrix} 0 & A_1 & & \\ & 0 & \ddots & \\ & & \ddots & A_{k-1} \\ & & & 0 \end{bmatrix},$$

we consider two k -by- k matrices

$$A' = \begin{bmatrix} 0 & \|A_1\| & & \\ & 0 & \ddots & \\ & & \ddots & \|A_{k-1}\| \\ & & & 0 \end{bmatrix} \quad \text{and} \quad A'' = \begin{bmatrix} 0 & m(A_1) & & \\ & 0 & \ddots & \\ & & \ddots & m(A_{k-1}) \\ & & & 0 \end{bmatrix},$$

where $\|\cdot\|$ and $m(\cdot)$ denote the operator norm and minimum modulus of a matrix, respectively. It is shown that the numerical radii $w(\cdot)$ of A , A' and A'' are related by the inequalities $w(A'') \leq w(A) \leq w(A')$. We also determine exactly when either of the inequalities becomes an equality.

Keywords: Numerical radius, block shift, minimum modulus.

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1. Introduction

An n -by- n complex matrix A is call a *block shift* if it is of the form

$$\begin{bmatrix} 0 & A_1 & & \\ & 0 & \ddots & \\ & & \ddots & A_{k-1} \\ & & & 0 \end{bmatrix},$$

where the A_j 's are in general rectangular matrices. In this paper, we obtain sharp upper and lower bounds for the numerical radius $w(A)$ of

*Corresponding author.

such an A . Recall that the *numerical radius* $w(B)$ of an n -by- n matrix B is the quantity

$$\max\{|\langle Bx, x \rangle| : x \in \mathbb{C}^n, \|x\| = 1\},$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the standard inner product and norm of vectors in \mathbb{C}^n , respectively. Note that $w(B)$ is the radius of the smallest circular disc centered at the origin which contains the *numerical range*

$$W(B) = \{\langle Bx, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$$

of B . For properties of the numerical range and numerical radius, the reader is referred to [3, Chapter 22] or [4, Chapter 1].

Note that if A is a block shift of the above form, then it is unitarily similar to $e^{i\theta}A$ for all real θ . Hence its numerical range is a closed circular disc centered at the origin with radius equal to its numerical radius. To estimate the latter, we consider two k -by- k scalar matrices

$$A' = \begin{bmatrix} 0 & \|A_1\| & & \\ & 0 & \ddots & \\ & & \ddots & \|A_{k-1}\| \\ & & & 0 \end{bmatrix} \quad \text{and} \quad A'' = \begin{bmatrix} 0 & m(A_1) & & \\ & 0 & \ddots & \\ & & \ddots & m(A_{k-1}) \\ & & & 0 \end{bmatrix},$$

where $\|A_j\|$ and $m(A_j)$, $1 \leq j \leq k-1$, are the operator norm and minimum modulus of A_j , respectively. Recall that the minimum modulus $m(B)$ of an m -by- n matrix B is, by definition, $\min\{\|Bx\| : x \in \mathbb{C}^n, \|x\| = 1\}$. In Sections 2 and 3 below, we show that $w(A'') \leq w(A) \leq w(A')$ always hold, and that, under the extra condition that the A_j 's are all nonzero (resp., under $A_1 \dots A_{k-1} \neq 0$), $w(A) = w(A')$ (resp., $w(A) = w(A'')$) implies that A' (resp., A'') is a direct summand of A (cf. Theorems 2.1 and 3.1). Examples are given showing that the nonzero conditions on the A_j 's are essential.

2. Upper bound

The main result of this section is the following theorem.

Theorem 2.1. *Let*

$$(2.1) \quad A = \begin{bmatrix} 0 & A_1 & & \\ & 0 & \ddots & \\ & & \ddots & A_{k-1} \\ & & & 0 \end{bmatrix} \quad \text{on} \quad \mathbb{C}^n = \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_k}$$

be an n -by- n block shift, where A_j is an n_j -by- n_{j+1} matrix for $1 \leq j \leq k-1$, and let

$$A' = \begin{bmatrix} 0 & \|A_1\| & & \\ & 0 & \ddots & \\ & & \ddots & \|A_{k-1}\| \\ & & & 0 \end{bmatrix} \quad \text{on } \mathbb{C}^k.$$

Then (a) $w(A) \leq w(A')$, and (b) under the assumption that $A_j \neq 0$ for all j , $w(A) = w(A')$ if and only if A is unitarily similar to $A' \oplus B$, where B is a block shift with $w(B) \leq w(A')$.

Proof. (a) Let $x = [x_1 \ \dots \ x_k]^T$ be a unit vector in \mathbb{C}^n such that $|\langle Ax, x \rangle| = w(A)$. Hence

$$\begin{aligned} w(A) &= \left| \left\langle \begin{bmatrix} 0 & A_1 & & \\ & 0 & \ddots & \\ & & \ddots & A_{k-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \right\rangle \right| \\ &= \left| \sum_{j=1}^{k-1} \langle A_j x_{j+1}, x_j \rangle \right| \\ &\leq \sum_{j=1}^{k-1} |\langle A_j x_{j+1}, x_j \rangle| \\ (2.2) \quad &\leq \sum_{j=1}^{k-1} \|A_j\| \|x_{j+1}\| \|x_j\| \\ &= \left\langle \begin{bmatrix} 0 & \|A_1\| & & \\ & 0 & \ddots & \\ & & \ddots & \|A_{k-1}\| \\ & & & 0 \end{bmatrix} \begin{bmatrix} \|x_1\| \\ \vdots \\ \|x_k\| \end{bmatrix}, \begin{bmatrix} \|x_1\| \\ \vdots \\ \|x_k\| \end{bmatrix} \right\rangle \\ (2.3) \quad &\leq w(A'), \end{aligned}$$

where the last inequality follows from the fact that $[\|x_1\| \ \dots \ \|x_k\|]^T$ is a unit vector in \mathbb{C}^k .

(b) Assume that $A_j \neq 0$ for all j , and that $w(A) = w(A')$. Then we have equalities throughout the chain of inequalities in (a). Since A' is an (entrywise) nonnegative matrix with irreducible real part, the equality in (2.3) yields, by [5, Proposition 3.3], that $x_j \neq 0$ for all j .

Let $\hat{x}_j = [0 \ \dots \ 0 \ x_j \ 0 \ \dots \ 0]^T$ for $1 \leq j \leq k$, and let K be the subspace of \mathbb{C}^n spanned by the \hat{x}_j 's. The equality in (2.2) implies that

$$(2.4) \quad |\langle A_j x_{j+1}, x_j \rangle| = \|A_j x_{j+1}\| \|x_j\| = \|A_j\| \|x_{j+1}\| \|x_j\|.$$

Hence $A_j x_{j+1} = a_j x_j$ for some scalar a_j . Therefore, $A\hat{x}_1 = 0$ and

$$A\hat{x}_j = [0 \ \dots \ 0 \ A_{j-1}x_j \ 0 \ \dots \ 0]^T = [0 \ \dots \ 0 \ a_{j-1}x_{j-1} \ 0 \ \dots \ 0]^T = a_{j-1}\hat{x}_{j-1}$$

(j-1)st (j-1)st

is in K for all j , $2 \leq j \leq k$. This shows that $AK \subseteq K$.

We next prove that $A^*K \subseteq K$. Indeed, we have $A^*\hat{x}_j = [0 \ \dots \ 0 \ A_j^*x_j \ 0 \ \dots \ 0]^T$
(j+1)st

for $1 \leq j \leq k-1$. Since

$$|a_j| \|x_j\|^2 = \|a_j x_j\| \|x_j\| = \|A_j x_{j+1}\| \|x_j\| = \|A_j\| \|x_{j+1}\| \|x_j\|$$

by (2.4), the nonzeroness of the A_j 's and x_j 's yields the same for the a_j 's. Letting $B_j = A_j/\|A_j\|$ and $y_j = (\|A_j\|/a_j)x_{j+1}$, we have $B_j y_j = (1/a_j)A_j x_{j+1} = x_j$ with $\|B_j\| = 1$ and

$$\|y_j\| = \frac{\|A_j\|}{|a_j|} \|x_{j+1}\| = \frac{\|A_j x_{j+1}\|}{|a_j|} = \|x_j\|$$

by (2.4). It follows from an extended lemma of Riesz and Sz.-Nagy that $B_j^* x_j = y_j$ (cf. [7, p. 215]). Therefore, we have $A_j^* x_j = (\|A_j\|^2/a_j)x_{j+1}$, which shows that $A_j^* \hat{x}_j = (\|A_j\|^2/a_j)\hat{x}_{j+1}$ is in K for $1 \leq j \leq k-1$. Moreover, we also have $A^* \hat{x}_k = 0$. Thus $A^*K \subseteq K$ as asserted.

Since $\{\hat{x}_j/\|x_j\|\}_{j=1}^k$ is an orthonormal basis of K , $A(\hat{x}_1/\|x_1\|) = 0$, and

$$\begin{aligned} A\left(\frac{\hat{x}_j}{\|x_j\|}\right) &= \frac{a_{j-1}\|x_{j-1}\|}{\|x_j\|} \frac{\hat{x}_{j-1}}{\|x_{j-1}\|} = \frac{a_{j-1}}{|a_{j-1}|} \frac{\|a_{j-1}x_{j-1}\|}{\|x_j\|} \frac{\hat{x}_{j-1}}{\|x_{j-1}\|} \\ &= \frac{a_{j-1}}{|a_{j-1}|} \frac{\|A_{j-1}x_j\|}{\|x_j\|} \frac{\hat{x}_{j-1}}{\|x_{j-1}\|} = \frac{a_{j-1}}{|a_{j-1}|} \|A_{j-1}\| \frac{\hat{x}_{j-1}}{\|x_{j-1}\|} \end{aligned}$$

for $2 \leq j \leq k$ by (2.4), we derive that the restriction $A|_K$ is unitarily similar to A' . Thus A is unitarily similar to $A' \oplus (A|_{K^\perp})$. We now show that $A|_{K^\perp}$ is also unitarily similar to a block shift. Indeed, let $\hat{H}_j = 0 \oplus \dots \oplus 0 \oplus \mathbb{C}^{n_j} \oplus 0 \oplus \dots \oplus 0$, $K_j = \mathbb{C}^{n_j} \ominus \bigvee \{x_j\}$, and $\hat{K}_j = 0 \oplus \dots \oplus 0 \oplus K_j \oplus 0 \oplus \dots \oplus 0$ for $1 \leq j \leq k$. Then $K^\perp = K_1 \oplus \dots \oplus K_k$. Since

$A\hat{H}_{j+1} \subseteq \hat{H}_j$ and $A^*\hat{x}_j \in \bigvee \{\hat{x}_{j+1}\}$ from before, we have $A\hat{K}_{j+1} \subseteq \hat{K}_j$ for $1 \leq j \leq k-1$. Moreover, $A\hat{H}_k = \{0\}$ implies that $A\hat{K}_k = \{0\}$. We conclude that $B \equiv A|_{K^\perp}$ is unitarily similar to a block shift with

$w(B) \leq w(A) = w(A')$. This proves one direction of (b). The converse is trivial. \square

Corollary 2.2. *Let A be an n -by- n block shift as in (2.1), and let $M = \max_j \|A_j\|$. Then*

- (a) $w(A) \leq M \cdot \cos(\pi/(k+1))$, and
- (b) $w(A) = M \cdot \cos(\pi/(k+1))$ if and only if A is unitarily similar to $(M \cdot J_k) \oplus B$, where B is a block shift with $w(B) \leq M \cdot \cos(\pi/(k+1))$.

Here J_k denotes the k -by- k Jordan block

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & \\ & & & & 0 \end{bmatrix},$$

whose numerical range is known to be $\{z \in \mathbb{C} : |z| \leq \cos(\pi/(k+1))\}$ (cf. [6]).

Proof of Corollary 2.2. (a) is an easy consequence of Theorem 2.1 (a) and [8, Lemma 5 (1)] while (b) follows from Theorem 2.1 (b) and [8, Lemma 5 (2)]. \square

We remark that the assertion in Theorem 2.1 (b) still holds for $n \leq 5$ even without the nonzero assumption on the A_j 's. This can be proven via a case-by-case verification by invoking, in most cases, the known result on the numerical ranges of square-zero matrices (cf. [9, Theorem 2.1]), which we omit. This is no longer the case for $n \geq 6$. Here we give a counterexample for $n = 6$.

Example 2.3. Let

$$A = \begin{bmatrix} 0 & \sqrt{2} & & & & \\ & 0 & 0 & & & \\ & & 0 & 1 & 0 & \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & & & 0 \end{bmatrix}$$

with $A_1 = [\sqrt{2}]$, $A_2 = [0]$, $A_3 = [1 \ 0]$ and $A_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$A' = \begin{bmatrix} 0 & \sqrt{2} & & & \\ & 0 & 0 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix},$$

and A and A' are unitarily similar to

$$\begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix},$$

respectively. Hence $w(A) = w(A') = \sqrt{2}/2$, but A' is not a direct summand of A . To see the latter, note that $\ker A \cap \ker A^* = \{0\}$. Hence A cannot have the 1-by-1 zero matrix $[0]$ as a direct summand, and thus A cannot be unitarily similar to $A' \oplus [0]$, or A' is not a direct summand of A .

3. Lower bound

Here is the main result of this section.

Theorem 3.1. *Let A be an n -by- n block shift as in (2.1), and let*

$$A'' = \begin{bmatrix} 0 & m(A_1) & & \\ & 0 & \ddots & \\ & & \ddots & m(A_{k-1}) \\ & & & 0 \end{bmatrix} \quad \text{on } \mathbb{C}^k.$$

Then (a) $w(A) \geq w(A'')$, and (b) under the assumption that $A_1 \dots A_{k-1} \neq 0$, $w(A) = w(A'')$ if and only if A is unitarily similar to $A'' \oplus C$, where C is a block shift with $w(C) \leq w(A'')$.

Our first lemma gives some basic properties of the minimum modulus of a rectangular matrix. For a square matrix (or, for that matter, an operator on a possibly infinite-dimensional Hilbert space), these appeared in [2, Theorem 1].

Lemma 3.2. *Let A be an m -by- n matrix. Then*

- (a) $m(A) > 0$ if and only if A is left invertible, and
- (b) $m(A)$ equals the minimum singular value of A . In particular, if $m < n$, then $m(A) = 0$.

Proof. (a) Note that $m(A) > 0$ means that there is a $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all x in \mathbb{C}^n , which is equivalent to the well-definedness of the linear transformation $Ax \mapsto x$ from the range of A to \mathbb{C}^n , or to the left-invertibility of A .

(b) Consider the polar decomposition of A : $A = V(A^*A)^{1/2}$, where V is an m -by- n partial isometry with $\ker V = \ker A$ (cf. [3, Problem 134]). Then

$$\begin{aligned} m(A) &= \min\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\} \\ &= \min\{\|V(A^*A)^{1/2}x\| : x \in \mathbb{C}^n, \|x\| = 1\} \\ &= \min\{\|(A^*A)^{1/2}x\| : x \in \mathbb{C}^n, \|x\| = 1\} \\ &= \text{minimum eigenvalue of } (A^*A)^{1/2} \\ &= \text{minimum singular value of } A. \end{aligned}$$

□

To prove Theorem 3.1 (b), we need another lemma to get around the restriction $A_1 \dots A_{k-1} \neq 0$.

Lemma 3.3. *If A_j is an n_j -by- n_{j+1} matrix, $1 \leq j \leq k-1$, such that $A_1 \dots A_{k-1} = 0$, then for any $\varepsilon > 0$, there are n_j -by- n_{j+1} matrices B_j such that $\|B_j - A_j\| < \varepsilon$ for all j and $B_1 \dots B_{k-1} \neq 0$.*

Proof. This is proven by induction on k . The case of $k = 2$ is trivial. We now assume that $k = 3$ and $A_1 A_2 = 0$. Consider the following four cases separately:

(i) $A_1 = 0$ and $A_2 = 0$. Let B_1 (resp., B_2) be the n_1 -by- n_2 (resp., n_2 -by- n_3) matrix with its $(1, 1)$ -entry equal to $\varepsilon/2$ and all other entries 0. Then $B_1 B_2$ has the $(1, 1)$ -entry $\varepsilon^2/4$, and hence is nonzero.

(ii) $A_1 \neq 0$ and $A_2 = 0$. Assume that a_{ij} , the (i, j) -entry of A_1 , is nonzero. Let $B_1 = A_1$ and let B_2 be the n_2 -by- n_3 matrix with its $(j, 1)$ -entry equal to $\varepsilon/2$ and all others 0. Then the $(i, 1)$ -entry of $B_1 B_2$ is $a_{ij}\varepsilon/2$, which is nonzero. Hence $B_1 B_2 \neq 0$.

(iii) $A_1 = 0$ and $A_2 \neq 0$. By symmetry, this case can be dealt with as in (ii).

(iv) $A_1, A_2 \neq 0$. Assume that x_i^T , the i th row of A_1 , and y_j , the j th column of A_2 , are nonzero. Since $x_i^T y_j = 0$, we may perturb y_j slightly to a column vector z_j such that $x_i^T z_j \neq 0$. Let $B_1 = A_1$ and B_2 be obtained from A_2 by replacing its y_j by z_j . Then $B_1 B_2 \neq 0$.

Note that in (ii) and (iv) above, we have actually shown that if $A_1 A_2 = 0$ and $A_1 \neq 0$, then for any $\varepsilon > 0$ there is a matrix B_2 such that $\|B_2 - A_2\| < \varepsilon$ and $A_1 B_2 \neq 0$. This will be used in the induction process below.

Assume that our assertion is true for $k-2$ and that $A_1 \dots A_{k-1} = 0$. If $A_1 \dots A_{k-2} = 0$, then the induction hypothesis implies, for each $\varepsilon > 0$, the existence of matrices B_1, \dots, B_{k-2} such that $\|B_j - A_j\| < \varepsilon$ for $1 \leq j \leq k-2$ and $B_1 \dots B_{k-2} \neq 0$. If $(B_1 \dots B_{k-2})A_{k-1} \neq 0$, then simply let $B_{k-1} = A_{k-1}$; otherwise, from (ii) and (iv) above, there is a matrix B_{k-1} such that $\|B_{k-1} - A_{k-1}\| < \varepsilon$ and $(B_1 \dots B_{k-2})B_{k-1} \neq 0$. On the other hand, if $A_1 \dots A_{k-2} \neq 0$, then, since $(A_1 \dots A_{k-2})A_{k-1} = 0$, (ii) and (iv) above yields a matrix B_{k-1} such that $\|B_{k-1} - A_{k-1}\| < \varepsilon$ and $(A_1 \dots A_{k-2})B_{k-1} \neq 0$. Letting $B_j = A_j$ for $1 \leq j \leq k-2$ proves our assertion. \square

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. (a) First assume that $A_1 \dots A_{k-1} \neq 0$. Since A'' is an (entrywise) nonnegative matrix, there is a unit vector $y = [y_1 \dots y_k]^T$ in \mathbb{C}^k with $y_j \geq 0$ for all j such that $\langle A'' y, y \rangle = w(A'')$ (cf. [5, Proposition 3.3]). Let u be a unit vector in \mathbb{C}^{n_k} such that $A_1 \dots A_{k-1} u \neq 0$, let $x_j = A_j A_{j+1} \dots A_{k-1} u / \|A_j A_{j+1} \dots A_{k-1} u\|$ for $1 \leq j \leq k-1$ and $x_k = u$, and let $v = [y_1 x_1 \dots y_k x_k]^T$. Then v is a unit vector in \mathbb{C}^n since

$$\|v\| = (|y_1|^2 \|x_1\|^2 + \dots + |y_k|^2 \|x_k\|^2)^{1/2} = (|y_1|^2 + \dots + |y_k|^2)^{1/2} = 1.$$

Moreover,

$$\langle Av, v \rangle = \sum_{j=1}^{k-1} \langle A_j (y_{j+1} x_{j+1}), y_j x_j \rangle = \sum_{j=1}^{k-1} y_{j+1} y_j \langle A_j x_{j+1}, x_j \rangle.$$

Note that

$$\begin{aligned} \langle A_j x_{j+1}, x_j \rangle &= \left\langle \frac{A_j A_{j+1} \dots A_{k-1} u}{\|A_{j+1} \dots A_{k-1} u\|}, \frac{A_j \dots A_{k-1} u}{\|A_j \dots A_{k-1} u\|} \right\rangle \\ &= \frac{\|A_j \dots A_{k-1} u\|}{\|A_{j+1} \dots A_{k-1} u\|} \geq m(A_j). \end{aligned}$$

Hence

$$(3.1) \quad \langle Av, v \rangle \geq \sum_{j=1}^{k-1} y_{j+1} y_j m(A_j) = \langle A'' y, y \rangle = w(A'').$$

It follows that $w(A) \geq w(A'')$ as asserted.

Now if $A_1 \dots A_{k-1} = 0$, then, for any $\varepsilon > 0$, let B_1, \dots, B_{k-1} be as in Lemma 3.3, and let

$$B = \begin{bmatrix} 0 & B_1 & & \\ & 0 & \ddots & \\ & & \ddots & B_{k-1} \\ & & & 0 \end{bmatrix} \text{ on } \mathbb{C}^n \text{ and } B'' = \begin{bmatrix} 0 & m(B_1) & & \\ & 0 & \ddots & \\ & & \ddots & m(B_{k-1}) \\ & & & 0 \end{bmatrix} \text{ on } \mathbb{C}^k.$$

From the first half of the proof, we have $w(B) \geq w(B'')$. Since

$$\|A'' - B''\| = \max_j |m(A_j) - m(B_j)| \leq \max_j \|A_j - B_j\| < \varepsilon$$

(cf. [10, Lemma 2.2 (1)]), we infer from the continuity of $w(\cdot)$ that $w(A) \geq w(A'')$ (cf. [3, Problem 220]). This completes the proof of (a).

(b) Assume that $A_1 \dots A_{k-1} \neq 0$ and $w(A) = w(A'')$. Let $y = [y_1 \dots y_k]^T \in \mathbb{C}^k$, $u \in \mathbb{C}^{n_k}$, $x_j \in \mathbb{C}^{n_j}$ for $1 \leq j \leq k$, and $v \in \mathbb{C}^n$ be as in the first half of the proof of (a). Let $\hat{x}_j = [0 \dots 0 \underset{j\text{th}}{x_j} 0 \dots 0]^T$ for

$1 \leq j \leq k$, and let K be the subspace of \mathbb{C}^n spanned by the \hat{x}_j 's. Since $A\hat{x}_1 = 0$ and

$$(3.2) \quad A\hat{x}_j = [0 \dots 0 \underset{(j-1)\text{st}}{\frac{A_{j-1}A_j \dots A_{k-1}u}{\|A_j \dots A_{k-1}u\|}} 0 \dots 0]^T = \frac{\|A_{j-1} \dots A_{k-1}u\|}{\|A_j \dots A_{k-1}u\|} \hat{x}_{j-1}, \quad 2 \leq j \leq k,$$

we obtain $AK \subseteq K$.

We next show that $A^*K \subseteq K$. Indeed, since $w(A) = w(A'')$, we have an equality in (3.1), which yields that $\|A_j \dots A_{k-1}u\| / \|A_{j+1} \dots A_{k-1}u\| = m(A_j)$ for all j , $1 \leq j \leq k-1$. This is because $A_j \neq 0$ for all j and thus A'' is a nonnegative matrix with irreducible real part, from which we infer that $y_j > 0$ for all j (cf. [5, Proposition 3.3]). Since the x_j 's are unit vectors satisfying $\|A_j x_{j+1}\| = m(A_j)$, we have $\langle (A_j^* A_j - m(A_j)^2 I_{n_{j+1}}) x_{j+1}, x_{j+1} \rangle = 0$, $1 \leq j \leq k-1$. From $A_j^* A_j \geq m(A_j)^2 I_{n_{j+1}}$, we infer that $A_j^* A_j x_{j+1} = m(A_j)^2 x_{j+1}$ and hence $A_j^* A_j A_{j+1} \dots A_{k-1} u = m(A_j)^2 A_{j+1} \dots A_{k-1} u$. This shows that $A_j^* x_j$ is a multiple of x_{j+1} and thus $A^* \hat{x}_j$ is a multiple of \hat{x}_{j+1} for $1 \leq j \leq k-1$. Therefore, $A^* \hat{x}_j$ is in K for all j , $1 \leq j \leq k-1$. Together with $A^* \hat{x}_k = 0$, these imply that $A^*K \subseteq K$. Hence A is unitarily similar to $(A|K) \oplus (A|K^\perp)$. Since $A\hat{x}_1 = 0$ and $A\hat{x}_j = m(A_{j-1})\hat{x}_{j-1}$, $2 \leq j \leq k$, from (3.2), we have the unitary similarity of $A|K$ and A'' . On the other hand, the unitary similarity of $A|K^\perp$ to a block shift follows as in the last part of the proof of Theorem 2.1 (b). \square

Corollary 3.4. *Let A be an n -by- n block shift as in (2.1), and let $m = \min_j m(A_j)$. Then*

- (a) $w(A) \geq m \cdot \cos(\pi/(k+1))$, and
- (b) $w(A) = m \cdot \cos(\pi/(k+1))$ if and only if A is unitarily similar to $(m \cdot J_k) \oplus C$, where C is a block shift with $w(C) \leq m \cdot \cos(\pi/(k+1))$.

This can be proven as Corollary 2.2 by using Theorem 3.1 and [8, Lemma 5].

Analogous to the situation in Section 2, Theorem 3.1 (b) remains true for $n \leq 3$ without the assumption $A_1 \dots A_{k-1} \neq 0$. This is no longer the case for $n \geq 4$. A counterexample for $n = 4$ is given below.

Example 3.5. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & \\ & 0 & 0 & 1 \\ & 0 & 0 & -1 \\ & & & 0 \end{bmatrix}$$

with $A_1 = [1 \ 1]$ and $A_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. In this case, $A_1 A_2 = [0]$ and

$$A'' = \begin{bmatrix} 0 & 0 & \\ & 0 & \sqrt{2} \\ & & 0 \end{bmatrix}.$$

Since $A^2 = 0$, we have $w(A) = \|A\|/2 = \sqrt{2}/2$ (cf. [9, Theorem 2.1]). On the other hand, we also have $w(A'') = \sqrt{2}/2$. But A'' is not a direct summand of A . This is because if it is, then A will be unitarily similar to $A'' \oplus [0]$, which is impossible since $\ker A \cap \ker A^* = \{0\}$.

A larger parameter than the minimum modulus of an m -by- n matrix A is its *reduced minimum modulus* $\gamma(A)$ defined by

$$\gamma(A) = \begin{cases} \min\{\|Ax\| : x \in \mathbb{C}^n, x \perp \ker A, \|x\| = 1\} & \text{if } A \neq 0, \\ 0 & \text{if } A = 0. \end{cases}$$

A general reference for $\gamma(A)$ (when A is an operator on a possibly infinite-dimensional Hilbert space) is [1]. For an n -by- n block shift A of the form (2.1), consider the k -by- k matrix

$$A''' = \begin{bmatrix} 0 & \gamma(A_1) & & \\ & 0 & \ddots & \\ & & \ddots & \gamma(A_{k-1}) \\ & & & 0 \end{bmatrix}.$$

We may expect to have $w(A''')$ as a lower bound for $w(A)$ under some extra conditions on A . The next theorem shows that this is indeed the case for small values of k .

Theorem 3.6. (i) Let $A = \begin{bmatrix} 0 & A_1 \\ 0 & 0 \end{bmatrix}$ and $A''' = \begin{bmatrix} 0 & \gamma(A_1) \\ 0 & 0 \end{bmatrix}$. Then

- (a) $w(A) \geq w(A''')$, and
- (b) $w(A) = w(A''')$ if and only if A is unitarily similar to $A''' \oplus \cdots \oplus A''' \oplus 0$.

(ii) Let

$$A = \begin{bmatrix} 0 & A_1 & \\ & 0 & A_2 \\ & & 0 \end{bmatrix} \text{ on } \mathbb{C}^n = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} \oplus \mathbb{C}^{n_3} \text{ and } A''' = \begin{bmatrix} 0 & \gamma(A_1) & \\ & 0 & \gamma(A_2) \\ & & 0 \end{bmatrix} \text{ on } \mathbb{C}^3.$$

Assume that $\text{rank } A_1 + \text{rank } A_2 > n_2$. Then

- (a) $w(A) \geq w(A''')$, and
- (b) $w(A) = w(A''')$ if and only if A is unitarily similar to $A''' \oplus C$, where C is a block shift with $w(C) \leq w(A''')$.

The proof is quite similar to the one for Theorem 3.1, which we omit. For larger values of k , the extra conditions on the A_j 's are two cumbersome to be of any practical use.

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(Hwa-Long Gau) DEPARTMENT OF MATHEMATICS, NATIONAL CENTRAL UNIVERSITY, CHUNGLI 32001, TAIWAN

E-mail address: hlgau@math.ncu.edu.tw

(Pei Yuan Wu) DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY, HSINCHU 30010, TAIWAN

E-mail address: pywu@math.nctu.edu.tw